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RESEARCH
REPORT
N° 8222
Février 2013
Project-Teams REO

ISSN 0249-6399   ISRN INRIA/RR--8222--FR+ENG

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Research Report n° 8222 — Février 2013 — 20 pages

Abstract: In this research report, we study the inverse problem of identifying a Robin coefficient defined on some non accessible part of the boundary from measurements available on the other part of the boundary, for (u,p) solution of the Stokes system. We prove a Lipschitz stability estimate, under the *a priori* assumption that the Robin coefficient is piecewise constant. To do so, we use unique continuation estimates for the Stokes system proved in [BEG12] and the approach developed by E. Sincich in [Sin07] to solve a similar inverse problem for the Laplace equation.

Key-words: Inverse problem, Lipschitz stability estimate, Stokes system.

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Estimation de stabilité Lipschitzienne pour le système de Stokes avec des conditions aux limites de type Robin

Résumé : Nous nous intéressons dans ce rapport de recherche à l'identification d'un coefficient de Robin défini sur une partie non accessible du bord à partir de mesures disponibles sur une autre partie du bord, pour (u, p) solution du système de Stokes. Nous prouvons une inégalité de stabilité Lipschitzienne sous l'hypothèse *a priori* que le coefficient de Robin est constant par morceaux. Pour ce faire, nous utilisons les estimations de continuation unique pour le système de Stokes prouvées dans [BEG12] et l'approche développée par E. Sincich dans [Sin07] pour résoudre un problème inverse similaire pour l'équation de Laplace.

Mots-clés : Problème inverse, Inégalité de stabilité Lipschitzienne, Système de Stokes.

1 Introduction

Let $\Omega \subset \mathbb{R}^d$, with $d = 2, 3$, be a Lipschitz bounded connected open set such that $\partial\Omega = \Gamma_l \cup \Gamma_0 \cup \Gamma_{out}$.

We are interested in the following system:

$$\begin{cases} -\Delta u + \nabla p &= 0, & \text{in } \Omega, \\ div\ u &= 0, & \text{in } \Omega, \\ u &= 0, & \text{in } \Gamma_l, \\ \dfrac{\partial u}{\partial n} - pn &= g, & \text{on } \Gamma_0, \\ \dfrac{\partial u}{\partial n} - pn + qu &= 0, & \text{on } \Gamma_{out}, \end{cases} \tag{1}$$

where $\Gamma_{out} = \bigcup_{i=1}^{N} \Gamma_i$ and n the exterior normal to Ω.

The aim of this paper is to identify the Robin coefficient q defined on the non accessible part of the boundary Γ_{out} from available data on Γ_0. We provide a Lipschitz stability estimate, under the *a priori* assumption that the Robin coefficient is piecewise constant on Γ_{out}:

$$q_{|\Gamma_i} = q_i \in \mathbb{R}_+, \text{ for } 1 \leq i \leq N. \tag{2}$$

Such problems can be viewed as a generalization of some problems which appear naturally in the modeling of biological problems like, for example, blood flow in the cardiovascular system (see [QV03] and [VCFJT06]) or airflow in the lungs (see [BGM10]). For an introduction on the modeling of the airflow in the lungs and on different boundary conditions which may be prescribed, we refer to [Egl12]. The part of the boundary Γ_0 represents a physical boundary on which measurements are available and Γ_{out} represents artificial boundaries on which Robin boundary conditions or mixed boundary conditions involving the fluid stress tensor and its flux at the outlet are prescribed.

The uniqueness for such problems is not an issue and has already been investigate in [BEG13]. It is obtained as a corollary of a unique continuation result for the Stokes system proven by C. Fabre and G. Lebeau (see [FL96]). Concerning the stability, logarithmic stability estimates have been obtained in [BEG13] and [BEG12] for more general than piecewise constant Robin coefficients. The main tools used in both cases are Carleman inequalities, global in the first case and local in the second one. Let us point out that, due to the mixed boundary conditions, the solution of system (1) is not regular in a neighborhood of the junction between two different boundary conditions. Thus, we can not expect in general that the solution belongs, at least, to $H^2(\Omega) \times H^1(\Omega)$. As a consequence, we can not use global Carleman inequalities requiring global regularity on the solution. The Lipschitz stability estimate obtained in this paper is based on local regularity on the solution of system (1) and on the open set Ω and on the stability estimates for the unique continuation property of the Stokes system proved in [BEG12].

Comparable problems have been widely studied for the Laplace equation (see for instance [ADPR03], [BCC08], [CFJL04], [CJ99], [CCL08] and [Sin07]. In this case, it is in general a problem arising in corrosion detection which consists of determining a Robin coefficient on the inaccessible portion of the boundary by electrostatic measurements performed on the accessible one. Most of the paper obtained logarithmic stability estimates but under some restricting assumption on the Robin coefficient and on the flux g, it is possible to obtain Lipschitz stability estimate. For instance, S. Chaabane and M. Jaoua obtained in [CJ99] both local and monotone global Lipschitz stability for regular Robin coefficient and under the assumption that the flux g is non negative. Relaxing this constraint, they obtained in [ADPR03] a logarithmic stability estimate. More recently, E. Sincich has obtained in [Sin07] a Lipschitz stability estimate under the further *a priori* assumption of a piecewise constant Robin coefficient.

Let us explain the structure of this paper. In Section 2, we begin with existence and regularity results on (u, p) solution of system (1): we state global and local regularity results which will be useful to prove the Lipschitz stability estimate. Then, we give in Section 3 technical lemmas and unique continuation estimates which will be useful to prove the Lipschitz stability estimate. Finally, Section 4 is dedicated to the main result: we state and prove the Lipschitz stability estimate.

In the following, we will not distinguish vector valued functions and scalar valued functions. Moreover, when we are not more specific, $C > 0$ is a constant whose value may change from a line to an other. Let us introduce some notations that we will be useful throughout this paper.

Notation 1.1. *For $x \in \mathbb{R}^d$ and $r > 0$, we denote $B_r(x)$ the ball of center x and of radius r.*

Notation 1.2. *Let $\Gamma \subset \partial\Omega$ be a non empty part of the boundary. We denote by*

$$\Gamma^{in} = \{x \in \Gamma / d(x, \overline{\partial\Omega\backslash\Gamma}) > 0\}.$$

Since the open set Ω is Lipschitz, it satisfies the cone property:

Definition 1.3 (Cone property)**.** *We say that Ω satisfies the cone property if there exists $\theta \in (0, \frac{\pi}{2})$ and $R_0 > 0$ such that for all $x_0 \in \partial\Omega$, there exists $\xi \in \mathbb{R}^d$, $|\xi| = 1$ such that the finite cone*

$$\mathcal{C} = \{x \in \mathbb{R}^d / (x - x_0) \cdot \xi > |x - x_0| \cos\theta, |x - x_0| \leq R_0\}$$

is included in Ω.

2 Regularity results

We focus in this section on global and local regularity results for system (1). Even if global $H^2 \times H^1$ regularity is not expected in general due to the mixed Dirichlet and Neumann boundary conditions, we obtain local regularity results inside the domain and near the boundary, as long as we stay away from the junction between two different boundary conditions.

We need to introduce functional spaces:

$$V_{\Gamma_l} = \{v \in H^1(\Omega) / v_{|\Gamma_l} = 0 \text{ and } div\ v = 0\}, \tag{3}$$

and

$$H_{\Gamma_l} = \overline{V}_{\Gamma_l}^{L^2(\Omega)}. \tag{4}$$

Moreover, for $g \in H^{-\frac{1}{2}}(\partial\Omega)$ and $v \in H^{\frac{1}{2}}(\partial\Omega)$, we denote by $< g, v >_{-\frac{1}{2},\frac{1}{2},\partial\Omega}$ the image of v by the linear form g.

2.1 Global regularity

Proposition 2.1. *Let $R_M > 0$, $\mathbb{1}_{\Gamma_0} g \in H^{-\frac{1}{2}}(\partial\Omega)$ and assume that q satisfies (2).*

Then, system (1) admits a unique solution $(u, p) \in V_{\Gamma_l} \times L^2(\Omega)$. Moreover, if we assume the $q \leq R_M$, there exists a constant $C(R_M) > 0$ such that:

$$\|u\|_{H^1(\Omega)} + \|p\|_{L^2(\Omega)} \leq C(R_M) \|\mathbb{1}_{\Gamma_0} g\|_{H^{-\frac{1}{2}}(\partial\Omega)}. \tag{5}$$

Proof of Proposition 2.1. The variational formulation of the problem is: find $u \in V_{\Gamma_l}$ such that for every $v \in V_{\Gamma_l}$,

$$\int_\Omega \nabla u : \nabla v + \int_{\Gamma_{out}} qu \cdot v = < \mathbb{1}_{\Gamma_0} g, v >_{-\frac{1}{2},\frac{1}{2},\partial\Omega} . \tag{6}$$

We denote by

$$a_q(u,v) = \int_\Omega \nabla u : \nabla v + \int_{\Gamma_{out}} qu \cdot v, \tag{7}$$

and

$$L(v) = < \mathbb{1}_{\Gamma_0} g, v >_{-\frac{1}{2},\frac{1}{2},\partial\Omega} .$$

We easily verify that a_q is a continuous symmetric bilinear form on V_{Γ_l}. According to Poincaré inequality, the bilinear form a_q is coercive on V_{Γ_l}. On the other hand, L is a continuous linear form on V_{Γ_l}. Thus we get the existence and uniqueness of $u \in V_{\Gamma_l}$ solution of equations (1) using the Lax-Milgram Theorem.We prove the existence and uniqueness of $p \in L^2(\Omega)$ in a classical way, by using De Rham Theorem and the Neumann boundary conditions. □

2.2 Local regularity

Inside the domain Ω, we have local regularity: this is resumed in Proposition 2.2. Moreover, locally near the boundary, as long as we stay away from the junction between two different boundary conditions, we can also obtain local regularity. We refer to Propositions 2.3 and 2.5 for a statement of these regularity results.

Proposition 2.2. *Let $R_M > 0$, $m \in \mathbb{N}^*$ and $\hat\omega \subset \Omega$ be a relatively compact open set. Let $\mathbb{1}_{\Gamma_0} g \in H^{-\frac{1}{2}}(\partial\Omega)$ and assume that q satisfies* (2)*.*

Then, the solution (u,p) of system (1) *belongs to $H^{m+1}(\hat\omega) \times H^m(\hat\omega)$. Moreover, if we assume that $q \le R_M$, there exists a constant $C(R_M) > 0$ such that:*

$$\|u\|_{H^{m+1}(\hat\omega)} + \|p\|_{H^m(\hat\omega)} \le C(R_M) \|\mathbb{1}_{\Gamma_0} g\|_{H^{-\frac{1}{2}}(\partial\Omega)}. \tag{8}$$

Although the proof of this result is classical, we give here a sketch of the proof for the sake of completeness.

Proof of Proposition 2.2. We prove this proposition by induction on m. For $m = 0$, the result is given by Proposition 2.1.

We assume that Proposition 2.2 holds for some fixed m. Let us prove that the proposition also holds for m + 1. Let ω be an open set of class $\mathcal{C}^{m+1,1}$ such that $\hat\omega \subset \omega \subset \Omega$. We localize in the neighborhood of $\hat\omega$. Let $\chi \in \mathcal{C}_c^\infty(\omega)$ be such that $\chi = 1$ in $\hat\omega$ and $0 \le \chi \le 1$ everywhere else. We denote by $(v,\pi) = (\chi u, \chi p)$. Note that (v,π) is solution if the following problem:

$$\begin{cases} -\Delta v + \nabla \pi &= -\Delta\chi u - 2\nabla u \nabla\chi + \nabla\chi p, & \text{in } \omega, \\ \text{div } v &= \nabla\chi \cdot u, & \text{in } \omega, \\ \frac{\partial v}{\partial n} - \pi n &= 0, & \text{on } \partial\omega. \end{cases}$$

Let us denote by $f = -\Delta\chi u - 2\nabla u\nabla\chi + \nabla\chi p$ and $h = \nabla\chi \cdot u$. By the induction assumption, we deduce that the (f,h) belongs to $H^m(\omega) \times H^{m+1}(\omega)$. By application of regularity result for the Stokes system with Neumann boundary condition (see [BF06]), we deduce that (v,π) belongs to $H^{m+2}(\omega) \times H^{m+1}(\omega)$. Since $\chi = 1$ on $\hat\omega$, we obtain the desired result. □

We now study the regularity near the boundary of the domain. Proposition 2.3 states regularity result in the restriction to Ω of a neighborhood of any point $x_j \in \Gamma_j^{in}$, for $i = j, \dots, N$.

Proposition 2.3. *Let $m \in \mathbb{N}^*$, $R_M > 0$, $1 \leq j \leq N$ and $x_j \in \Gamma_j^{in}$. We assume that Γ_j is of class $\mathcal{C}^{m,1}$ if $m \geq 1$ and Lipschitz otherwise. Let $\mathbb{1}_{\Gamma_0} g \in H^{-\frac{1}{2}}(\partial\Omega)$ and assume that q satisfies (2).*

Then, there exists $R > 0$ such that the solution (u,p) of system (1) belongs to $H^{m+1}(B_R(x_j) \cap \Omega) \times H^m(B_R(x_j) \cap \Omega)$. Furthermore, if we assume that $q \leq R_M$, there exists a constant $C(R_M) > 0$ such that

$$\|u\|_{H^{m+1}(B_R(x_j)\cap\Omega)} + \|p\|_{H^m(B_R(x_j)\cap\Omega)} \leq C(R_M) \|\mathbb{1}_{\Gamma_0} g\|_{H^{-\frac{1}{2}}(\partial\Omega)}.$$

The proof of Proposition 2.3 relies on classical argument and is similar to the proof of Proposition 2.2.

Then, we deduce from Proposition 2.3 local Hölder regularity on (u,p) solution of system (1) near the boundary Γ_j, as long as we stay away from the junction between two different boundary conditions.

Corollary 2.4. *Let $1 \leq j \leq N$ and $x_j \in \Gamma_j^{in}$. We assume that Γ_j is of class $\mathcal{C}^{2,1}$. Let $\mathbb{1}_{\Gamma_0} g \in H^{-\frac{1}{2}}(\partial\Omega)$ and assume that q satisfies (2).*

Then, there exists $R > 0$ and $0 < \beta < 1$ such that the solution (u,p) of system (1) belongs to $\mathcal{C}^{1,\beta}(\overline{B_R(x_j) \cap \Omega}) \times \mathcal{C}^{0,\beta}(\overline{B_R(x_j) \cap \Omega})$ for all $j = 1, \ldots, N$.

Proof of Corollary 2.4. Let $1 \leq j \leq N$. Thanks to Proposition 2.3, there exists $R_j > 0$ such that $(u,p) \in H^3(B_{R_j}(x_j) \cap \Omega) \times H^2(B_{R_j}(x_j) \cap \Omega)$. Thanks to Proposition 2.2, we know that there exists a connected open set $\tilde{\Omega} \subset \Omega$ of class $\mathcal{C}^{2,1}$ such that $\displaystyle\bigcup_{j=1}^{N} \left(B_{R_j}(x_j) \cap \Omega\right) \subset \tilde{\Omega}$ and $(u,p) \in H^3(\tilde{\Omega}) \times H^2(\tilde{\Omega})$. Then, since for all $m \in \mathbb{N}$ such that $2(m-1) \leq d < 2m$, there exists $0 < \lambda < 1$ such that $H^m(\tilde{\Omega}) \hookrightarrow \mathcal{C}^{0,\lambda}(\overline{\tilde{\Omega}})$ (see [Ada75]), we deduce that there exits $0 < \beta < 1$ such that $H^2(\tilde{\Omega}) \subset \mathcal{C}^{0,\beta}(\overline{\tilde{\Omega}})$, which implies that $(u,p) \in \mathcal{C}^{1,\beta}(\overline{B_R(x_j) \cap \Omega}) \times \mathcal{C}^{0,\beta}(\overline{B_R(x_j) \cap \Omega})$ for all $j = 1, \ldots, N$, with $R = \min_{1 \leq j \leq N} R_j$. ☐

Proposition 2.5 below states regularity result in the restriction to Ω of a neighborhood of any point $x_0 \in \Gamma_0^{in}$.

Proposition 2.5. *Let $R_M > 0$, $m \in \mathbb{N}$ and $x_0 \in \Gamma_0^{in}$. We assume that Γ_0 is of class $\mathcal{C}^{m,1}$ if $m \geq 1$ and Lipschitz otherwise. Let $\mathbb{1}_{\Gamma_0} g \in H^{m-\frac{1}{2}}(\partial\Omega)$ and assume that q satisfies (2).*

Then, there exists $R > 0$ such that the solution (u,p) of system (1) belongs to $H^{m+1}(B_R(x_0) \cap \Omega) \times H^m(B_R(x_0) \cap \Omega)$. Furthermore, if we assume that $q \leq R_M$, there exists a constant $C(R_M) > 0$ such that

$$\|u\|_{H^{m+1}(B_R(x_0)\cap\Omega)} + \|p\|_{H^m(B_R(x_0)\cap\Omega)} \leq C(R_M) \|\mathbb{1}_{\Gamma_0} g\|_{H^{m-\frac{1}{2}}(\partial\Omega)}.$$

3 Preliminary results

3.1 Useful lemmas

The following lemmas will be useful throughout this paper.

Lemma 3.1. *Let $A > 0$, $B > 0$, $C_1 > 0$, $C_2 > 0$ and $D > 0$. We assume that there exists $\gamma_1 > 0$ such that*

$$D \leq A e^{C_1 \gamma} + B e^{-C_2 \gamma}, \tag{9}$$

for all $\gamma \geq \gamma_1$ and $c_0 > 0$ such that $D \leq c_0 B$. Then, there exists $C > 0$ such that:

$$D \leq C A^{\frac{C_2}{C_1 + C_2}} B^{\frac{C_1}{C_1 + C_2}}.$$

We refer to [Rob95] for a proof of this lemma.

Lemma 3.2. *Let $A \in \mathbb{R}$, $\mu \in \mathbb{R}^*$ and $(\beta_k)_{k\in\mathbb{N}} \in \mathbb{R}^{\mathbb{N}}$. If, for all $k \in \mathbb{N}^*$, we have*

$$\beta_k \le \frac{1}{\mu^{k-1}} \left(\beta_{k-1}\right)^{\alpha} A^{1-\alpha}, \tag{10}$$

then

$$\beta_k \le \frac{1}{\mu^{\iota}} \beta_0^{\alpha^k} A^{1-\alpha^k},$$

where $\iota = \sum_{j=1}^{k-1} j\alpha^{k-1-j}$.

This lemma is proved in [BD10]. For the sake of completeness, we write it.

Proof of Lemma 3.2. We rewrite inequality (10) under the form:

$$\frac{\beta_k}{A} \le \frac{1}{\mu^{k-1}} \left(\frac{\beta_{k-1}}{A}\right)^{\alpha}.$$

By iterating the above inequality, we get:

$$\frac{\beta_k}{A} \le \frac{1}{\mu^{\iota}} \left(\frac{\beta_0}{A}\right)^{\alpha^k},$$

where $\iota = \sum_{j=1}^{k-1} j\alpha^{k-1-j}$. □

Lemma 3.3. *Let A, B, C_1 and D be positive numbers and $0 < \alpha < 1$. Assume that*

$$D \le C_1 A^{\alpha} B^{1-\alpha}.$$

Then, for all $\epsilon > 0$

$$D \le \frac{c}{\epsilon} A + \epsilon^s B,$$

where $c = C_1^{\frac{1}{\alpha}}$ and $s = \dfrac{\alpha}{1-\alpha}$.

Proof of Lemma 3.3. Let $\epsilon > 0$. We rewrite $C_1 A^{\alpha} B^{1-\alpha} = \left(\frac{C_1^{1/\alpha}}{\epsilon} A\right)^{\alpha} \epsilon^{\alpha} B^{1-\alpha}$. Then it is sufficient to apply Young inequality:

$$D \le \alpha \frac{C_1^{1/\alpha}}{\epsilon} A + (1-\alpha)\epsilon^s B.$$

Since $0 < \alpha < 1$, the desired inequality follows. □

3.2 Unique continuation estimates

In this section, we state some unique continuation estimates for the Stokes system which will be useful in the next subsection to prove the Lipschitz stability estimate. They are obtained as corollaries of unique continuation estimates proved in [BEG12]. Let us begin by recalling a proposition from [BEG12] which allows to transmit information from a part of the boundary of Ω to a relatively compact open set in Ω.

Proposition 3.4. *Assume that D is of class $\mathcal{C}^\infty$. Let $0 < \nu \leq \frac{1}{2}$. Let Γ be a non empty open subset of the boundary of D. Let $\hat{\omega}$ be a relatively compact open set in D. Then, there exists $C, \sigma > 0$, such that for all $\epsilon > 0$ and for all $(u,p) \in H^{\frac{3}{2}+\nu}(D) \times H^{\frac{3}{2}+\nu}(D)$ solution of*

$$\begin{cases} -\Delta u + \nabla p &= 0, \quad \text{in } D, \\ \text{div } u &= 0, \quad \text{in } D, \end{cases} \tag{11}$$

$$\|u\|_{H^1(\hat{\omega})} + \|p\|_{H^1(\hat{\omega})} \leq \frac{C}{\epsilon} \left(\|u\|_{H^1(\Gamma)} + \|p\|_{H^1(\Gamma)} + \left\| \frac{\partial u}{\partial n} \right\|_{L^2(\Gamma)} + \left\| \frac{\partial p}{\partial n} \right\|_{L^2(\Gamma)} \right) + \epsilon^\sigma (\|u\|_{H^1(D)} + \|p\|_{H^1(D)}).$$

In our case, we can not apply directly Proposition 3.4 in the open set Ω to (u,p) solution of system (1) because regularity is needed both on the solution of the Stokes system (11) and on the open set. Nevertheless, Proposition 3.5 below is obtain as a corollary of Proposition 3.4 and allows us to transmit information from a part of the boundary $\Gamma \subset \Gamma_0$ to a relatively compact open $\hat{\omega}$ set included in Ω.

Proposition 3.5. *Assume that Γ_0 is of class $\mathcal{C}^\infty$. Let $R_M > 0$, $M_1 > 0$, $\Gamma \subseteq \Gamma_0$ be a non empty open subset of the boundary of Ω such that $\left(\overline{\Gamma} \cap \overline{\Gamma}_l\right) \bigcup \left(\overline{\Gamma} \cap \overline{\Gamma}_{out}\right) = \emptyset$ and $\hat{\omega} \subset \Omega$ be a relatively compact open set. Let $\mathbb{1}_{\Gamma_0} g \in H^{\frac{3}{2}}(\partial\Omega)$ be such that $\|\mathbb{1}_{\Gamma_0} g\|_{H^{\frac{3}{2}}(\partial\Omega)} \leq M_1$ and assume that q satisfies (2) and $q \leq R_M$.*

Then, there exist constants $C(R_M, M_1) > 0$ and $0 < \delta < 1$ such that for all (u,p) solution of system (1), the following inequality is satisfied:

$$\|u\|_{H^1(\hat{\omega})} + \|p\|_{H^1(\hat{\omega})} \leq C(R_M, M_1) \left(\|u\|_{L^2(\Gamma)} + \|p\|_{L^2(\Gamma)} + \left\| \frac{\partial u}{\partial n} \right\|_{L^2(\Gamma)} + \left\| \frac{\partial p}{\partial n} \right\|_{L^2(\Gamma)} \right)^\delta. \tag{12}$$

Proof of Proposition 3.5. Thanks to local regularity results stated in Subsection 2.2, we know that there exists a connected open set $\tilde{\Omega} \subset \Omega$ of class $\mathcal{C}^\infty$ such that $\Gamma \subset \partial\tilde{\Omega}$, $\hat{\omega} \subset \tilde{\Omega}$ and such that the solution (u,p) of (1) belongs to $H^3(\tilde{\Omega}) \times H^2(\tilde{\Omega})$. Moreover, there exists a constant $C(R_M, M_1) > 0$ such that:

$$\|u\|_{H^3(\tilde{\Omega})} + \|p\|_{H^2(\tilde{\Omega})} \leq C(R_M, M_1). \tag{13}$$

We apply Proposition 3.4: there exists $\sigma > 0$ and $C > 0$ such that, for all $\tilde{\epsilon} > 0$,

$$\begin{aligned} &\|u\|_{H^1(\hat{\omega})} + \|p\|_{H^1(\hat{\omega})} \\ &\leq \frac{C}{\tilde{\epsilon}} \left(\|u\|_{H^1(\Gamma)} + \|p\|_{H^1(\Gamma)} + \left\| \frac{\partial u}{\partial n} \right\|_{L^2(\Gamma)} + \left\| \frac{\partial p}{\partial n} \right\|_{L^2(\Gamma)} \right) + \tilde{\epsilon}^\sigma \left(\|u\|_{H^1(\tilde{\Omega})} + \|p\|_{H^1(\tilde{\Omega})} \right). \end{aligned} \tag{14}$$

Note that it is the H^1 norms of u and p on Γ which appear in the first term in the right hand-side of (14). In order to replace them with the L^2 norms of u and p on Γ, we use an interpolation inequality: there exists $c > 0$ such that

$$\|u\|_{H^1(\Gamma)} + \|p\|_{H^1(\Gamma)} \leq c \left(\|u\|^{\frac{1}{3}}_{L^2(\Gamma)} \|u\|^{\frac{2}{3}}_{H^{\frac{3}{2}}(\Gamma)} + \|p\|^{\frac{1}{3}}_{L^2(\Gamma)} \|p\|^{\frac{2}{3}}_{H^{\frac{3}{2}}(\Gamma)} \right).$$

Let $\overline{\epsilon} > 0$. If we write

$$\|u\|^{\frac{1}{3}}_{L^2(\Gamma)} \|u\|^{\frac{2}{3}}_{H^{\frac{3}{2}}(\Gamma)} = \left(\frac{1}{\overline{\epsilon}} \|u\|_{L^2(\Gamma)} \right)^{\frac{1}{3}} \left(\overline{\epsilon}^{\frac{1}{2}} \|u\|_{H^{\frac{3}{2}}(\Gamma)} \right)^{\frac{2}{3}},$$

and

$$\|p\|_{L^2(\Gamma)}^{\frac{1}{3}} \|p\|_{H^{\frac{3}{2}}(\Gamma)}^{\frac{2}{3}} = \left(\frac{1}{\overline{\epsilon}} \|p\|_{L^2(\Gamma)} \right)^{\frac{1}{3}} \left(\overline{\epsilon}^{\frac{1}{2}} \|p\|_{H^{\frac{3}{2}}(\Gamma)} \right)^{\frac{2}{3}},$$

according to Young inequality and to the trace injection $H^2(\tilde{\Omega}) \hookrightarrow H^{\frac{3}{2}}(\Gamma)$, we obtain:

$$\|u\|_{H^1(\Gamma)} + \|p\|_{H^1(\Gamma)} \leq c \left(\overline{\epsilon}^{\frac{1}{2}} \left(\|u\|_{H^3(\tilde{\Omega})} + \|p\|_{H^2(\tilde{\Omega})} \right) + \frac{1}{\overline{\epsilon}} \left(\|u\|_{L^2(\Gamma)} + \|p\|_{L^2(\Gamma)} \right) \right). \tag{15}$$

Let $\epsilon > 0$. By combining inequalities (15) with $\overline{\epsilon} = \tilde{\epsilon}^{2(\sigma+1)}$ and inequalities (14) with $\tilde{\epsilon} = \epsilon^{\frac{1}{2\sigma+3}}$, we obtain the existence of $C > 0$ and $s > 0$ such that for all $\epsilon > 0$:

$$\begin{aligned} &\|u\|_{H^1(\hat{\omega})} + \|p\|_{H^1(\hat{\omega})} \\ &\leq C \left(\frac{1}{\epsilon} \left(\|u\|_{L^2(\Gamma)} + \|p\|_{L^2(\Gamma)} + \left\| \frac{\partial u}{\partial n} \right\|_{L^2(\Gamma)} + \left\| \frac{\partial p}{\partial n} \right\|_{L^2(\Gamma)} \right) + \epsilon^s \left(\|u\|_{H^3(\tilde{\Omega})} + \|p\|_{H^2(\tilde{\Omega})} \right) \right). \end{aligned} \tag{16}$$

We conclude by using Lemma 3.1 and inequality (13). □

Then, Lemma 3.6 allows to transmit information inside the domain Ω, from a relatively compact open set include in Ω to an other.

Lemma 3.6. *[Three balls inequality] Let $\rho > 0$ and $q \in \mathbb{R}^d$. There exist $C > 0$, $0 < \alpha < 1$ such that for all function $(u, p) \in H^1(B_{8\rho}(q)) \times H^1(B_{8\rho}(q))$ solution of*

$$\begin{cases} -\Delta u + \nabla p &= 0, \\ div\ u &= 0, \end{cases} \tag{17}$$

in $B_{8\rho}(q)$ the following inequality is satisfied:

$$\begin{aligned} &\|u\|_{H^1(B_{3\rho}(q))} + \|p\|_{L^2(B_{3\rho}(q))} \\ &\qquad \leq C \left(\|u\|_{H^1(B_\rho(q))} + \|p\|_{L^2(B_\rho(q))} \right)^\alpha \left(\|u\|_{H^1(B_{8\rho}(q))} + \|p\|_{L^2(B_{8\rho}(q))} \right)^{1-\alpha}, \end{aligned} \tag{18}$$

with $\alpha = \dfrac{g(3\rho) - g\left(\frac{7}{2}\rho\right)}{g(\frac{\rho}{3}) - g\left(\frac{7}{2}\rho\right)}$ and with $g(r) = e^{-\lambda r^2}$, for λ large enough.

In the following, in order to refer to this inequality, we will say that the three balls inequality associated to q and ρ is satisfied, with the associated constants $C > 0$ and $\alpha > 0$.

Remark 3.7. *Lemma 3.6 is the counterpart, in the case of the Stokes system, of the so-called three balls inequality for the Laplacian. We refer to [BD10] or [LNW08] for a three balls inequality for the Laplacian. Note that in [LUW10], C.-H. Lin, G. Uhlmann and J.-N. Wang have obtained an optimal three balls inequality for the Stokes system involving only the velocity in the L^2 norm. From this inequality, they derive an upper bound on the vanishing order of any non trivial solution u to the Stokes system.*

The ideas of the proof are the same as those developed in [BEG12]. We refer to [Egl12] for a complete proof of this result. In the proof of the Lipschitz stability estimate, we will apply Lemma 3.6 to a sequence of balls with decreasing radius which approachs the boundary. To this aim, we need to know the behavior of the constants when we pass from a ball to another. This is done in Lemma 3.8 below.

Lemma 3.8. *Let $\rho > 0$, $(\overline{q}, q) \in \mathbb{R}^d \times \mathbb{R}^d$, and $\mu \in (0,1)$. We denote by $\rho = \mu\overline{\rho}$. We assume that the three balls inequality (18) associated to $\overline{q}$ and $\overline{\rho}$ holds for some constants $C > 0$ and $\alpha > 0$.*

Then, for all functions $(u,p) \in H^1(B_{8\rho}(q)) \times H^1(B_{8\rho}(q))$ solution of (17) in $B_{8\rho}(q)$, the following inequality is satisfied:

$$\begin{aligned}\|u\|_{H^1(B_{3\rho}(q))} &+ \|p\|_{L^2(B_{3\rho}(q))} \\ &\leq \tilde{C}\left(\|u\|_{H^1(B_{\rho}(q))} + \|p\|_{L^2(B_{\rho}(q))}\right)^{\alpha}\left(\|u\|_{H^1(B_{8\rho}(q))} + \|p\|_{L^2(B_{8\rho}(q))}\right)^{1-\alpha}, \quad (19)\end{aligned}$$

where $\tilde{C} = \dfrac{C}{\mu} > 0$. In other words, the three balls inequality associated to q and ρ is satisfied with the associated constants $\dfrac{C}{\mu} > 0$ and $\alpha > 0$.

Proof of Lemma 3.8. This lemma is inspired from [BD10] where L. Bourgeois and J. Dardé are concerned with the operator $P_k = -\Delta - k$, with $k \in \mathbb{R}$, and use similar techniques.

Let $m \in \mathbb{N}^*$. By performing the change of variables

$$\begin{array}{rcl} B_{m\overline{\rho}}(\overline{q}) & \to & B_{m\rho}(q), \\ x & \to & q + (x - \overline{q})\mu, \end{array}$$

we get:

$$\int_{B_{m\rho}(q)} |u(x)|^2 + |\nabla u(x)|^2 dx = \mu^d \int_{B_{m\overline{\rho}}(\overline{q})} |u(q + \mu(x - \overline{q}))|^2 + |\nabla u(q + \mu(x - \overline{q}))|^2 dx.$$

Let us denote by $\overline{u}(x) = u(q + \mu(x - \overline{q}))$ and $\overline{p}(x) = \mu p(q + \mu(x - \overline{q}))$. Noticing that $\nabla\overline{u}(x) = \mu\nabla u(q + \mu(x - \overline{q}))$, we get:

$$\int_{B_{m\rho}(q)} |u(x)|^2 + |\nabla u(x)|^2 dx = \mu^d \int_{B_{m\overline{\rho}}(\overline{q})} |\overline{u}(x)|^2 dx + \frac{1}{\mu^2}|\nabla\overline{u}(x)|^2 dx.$$

Moreover, since

$$\int_{B_{m\rho}(q)} |p(x)|^2 dx = \mu^{d-2} \int_{B_{m\overline{\rho}}(\overline{q})} |\overline{p}(x)|^2 dx,$$

and since $0 < \mu < 1$, we obtain:

$$\begin{aligned}\mu^{\frac{d}{2}} &\left(\|\overline{u}\|_{H^1(B_{m\overline{\rho}}(\overline{q}))} + \|\overline{p}\|_{L^2(B_{m\overline{\rho}}(\overline{q}))}\right) \\ &\leq \|u\|_{H^1(B_{m\rho}(q))} + \|p\|_{L^2(B_{m\rho}(q))} \\ &\leq \mu^{\frac{d}{2}-1}\left(\|\overline{u}\|_{H^1(B_{m\overline{\rho}}(\overline{q}))} + \|\overline{p}\|_{L^2(B_{m\overline{\rho}}(\overline{q}))}\right).\end{aligned}$$

Observe that $(\overline{u}, \overline{p})$ is solution in $B_{8\overline{\rho}}(\overline{q})$ of system (17):

$$-\Delta\overline{u}(\overline{x}) + \nabla\overline{p}(\overline{x}) = \mu^2\left(-\Delta u(x) + \nabla p(x)\right) = 0,$$

for $\overline{x} \in B_{8\overline{\rho}}(\overline{q})$ and where $x = q + (\overline{x} - \overline{q})\mu \in B_{8\rho}(q)$.

Thus, $(\overline{u}, \overline{p})$ satisfies (18) for $C > 0$ and $\alpha > 0$. We deduce that:

$$\begin{aligned}\|u\|_{H^1(B_{3\rho}(q))} + \|p\|_{L^2(B_{3\rho}(q))} &\leq \mu^{\frac{d}{2}-1}\left(\|\overline{u}\|_{H^1(B_{3\overline{\rho}}(\overline{q}))} + \|\overline{p}\|_{L^2(B_{3\overline{\rho}}(\overline{q}))}\right) \\ &\leq \mu^{\frac{d}{2}-1} C\left(\|\overline{u}\|_{H^1(B_{\overline{\rho}}(\overline{q}))} + \|\overline{p}\|_{L^2(B_{\overline{\rho}}(\overline{q}))}\right)^{\alpha}\left(\|\overline{u}\|_{H^1(B_{8\overline{\rho}}(\overline{q}))} + \|\overline{p}\|_{L^2(B_{8\overline{\rho}}(\overline{q}))}\right)^{1-\alpha} \\ &\leq \frac{C}{\mu}\left(\|u\|_{H^1(B_{\rho}(q))} + \|p\|_{L^2(B_{\rho}(q))}\right)^{\alpha}\left(\|u\|_{H^1(B_{8\rho}(q))} + \|p\|_{L^2(B_{8\rho}(q))}\right)^{1-\alpha}.\end{aligned}$$

□

4 Main result

Theorem 4.1. *Assume that Γ_0 is of class $\mathcal{C}^\infty$ and Γ_i is of class $\mathcal{C}^{2,1}$ for $i = 1, \ldots, N$. Let $m > 0$, $R_M > 0$, $M_1 > 0$, $\Gamma \subseteq \Gamma_0$ be a non empty open subset of the boundary of Ω such that $\left(\overline{\Gamma} \cap \overline{\Gamma}_l\right) \bigcup \left(\overline{\Gamma} \cap \overline{\Gamma}_{out}\right) = \emptyset$ and let $g \in H^{\frac{3}{2}}(\Gamma_0)$ be non identically zero on Γ_0 and such that $\|g\|_{H^{\frac{3}{2}}(\Gamma_0)} \leq M_1$. We assume that q_k satisfies (2) with $q_i = q_i^k$ be such that $q_i^k \leq R_M$ for $i = 1, \ldots, N$ and $k = 1, 2$. Let us denote by (u_k, p_k) the solution of (1) associated to $q = q^k$ for k=1,2. We assume that there exists $x_j \in \Gamma_j^{in}$ such that $|u_2(x_j)| > m$, for all $j = 1, \ldots, N$.*

Then, there exists $C(R_M, M_1, N, m) > 0$ such that

$$\|q^1 - q^2\|_{L^\infty(\Gamma_{out})} \leq C(R_M, M_1, N, m) \left(\|u_1 - u_2\|_{L^2(\Gamma)} + \|p_1 - p_2\|_{L^2(\Gamma)} + \left\| \frac{\partial p_1}{\partial n} - \frac{\partial p_2}{\partial n} \right\|_{L^2(\Gamma)} \right). \quad (20)$$

Remark 4.2. *Since g is not identically zero on Γ_0, we know, thanks to the uniqueness result (see [BEG13]), that for all $j = 1, \ldots, N$, there exists $x_j \in \Gamma_j^{in}$ such that $u_2(x_j) \neq 0$. We notice however that the constant involved in the estimate (20) depends on u_2 through the constant m. Finding a uniform lower bound for a solution u of system (1) remains an open question.*

Remark 4.3. *Let $1 \leq i \leq N$ and $x_i \in \Gamma_i^{in}$ be such that $|u_2(x_i)| > m$. Let us give an idea of how information goes from Γ to a neighborhood $\mathcal{V}_i$ of x_i. Thanks to Proposition 3.5, information goes from Γ to a relatively compact open set B_0 included in Ω. Then, we use a sequence of balls $(B_k)_{k\in\mathbb{N}}$ with decreasing radius to approach the boundary near x_i, taking into account Lemma 3.8. Finally, we use the boundary condition on Γ_i and the local Hölder regularity of the solution on $\mathcal{V}_i$ (see Corollary 2.4). We refer to Figure 1 for an illustration.*

Proof of Theorem 4.1. We follow the approach developed in [Sin07] in the case of the Laplace equation.

We consider:

$$(w, \pi) = \left(\frac{u_1 - u_2}{\sum_{j=1}^N |q_j^1 - q_j^2|}, \frac{p_1 - p_2}{\sum_{j=1}^N |q_j^1 - q_j^2|} \right). \quad (21)$$

According to Proposition 2.1, (w, π) belongs to $V_{\Gamma_l} \times L^2(\Omega)$. Since for $k = 1, 2$, q^k is piecewise constant, (w, π) is solution of:

$$\left\{ \begin{array}{rcll} -\Delta w + \nabla \pi & = & 0, & \text{in } \Omega, \\ \text{div } w & = & 0, & \text{in } \Omega, \\ w & = & 0, & \text{on } \Gamma_l, \\ \dfrac{\partial w}{\partial n} - \pi n & = & 0, & \text{on } \Gamma_0, \\ \dfrac{\partial w}{\partial n} - \pi n + q^1 w & = & \dfrac{(q^2 - q^1)}{\sum_{j=1}^N |q_j^1 - q_j^2|} u_2, & \text{on } \Gamma_{out}. \end{array} \right. \quad (22)$$

Observe that if we do not assume that q^k is piecewise constant, we get additional terms which depend on the derivative of q_k in system (22).

<u>**Step 1**</u> : Since the open set Ω satisfies the cone property, there exists $\theta \in (0, 1)$ and $R_0 > 0$ such that for all $i = 1, \ldots, N$ there exists $\xi_i \in \mathbb{R}^d$, $|\xi_i| = 1$ such that the finite cone $\mathcal{C}_i = \{x \in \mathbb{R}^d / (x - x_i) \cdot \xi_i > |x - x_i| \cos\theta$ and $|x - x_i| \leq R_0\}$ is included in Ω.

We are going to construct a sequence of balls $(B_{\rho_k}(\zeta_{i,k}))_{k\in\mathbb{N}}$ with decreasing radius and whose center is converging through x_i.

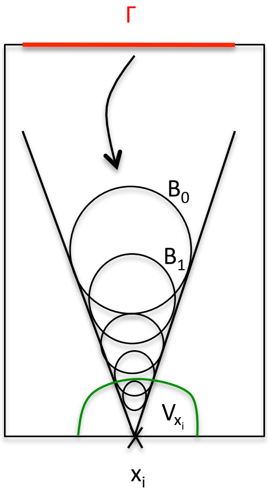

Figure 1: Figure illustrating how informations spread in the proof of Theorem 4.1.

For this sequence of balls, we will prove that there exists $0 < \alpha < 1$, $0 < \mu < 1$ and $C > 0$ such that for all $i = 1, \dots, N$ and $k \in \mathbb{N}$, the following estimate is satisfied for all $\epsilon > 0$:

$$\begin{aligned} &\|w\|_{H^1(B_{\rho_k}(\zeta_{i,k}))} + \|\pi\|_{L^2(B_{\rho_k}(\zeta_{i,k}))} \\ &\leq \frac{e^{\frac{C}{\alpha^k}\log\left(\frac{1}{\mu^{k-1}}\right)}}{\epsilon}\left(\|w\|_{H^1(B_{\rho_0}(\zeta_{i,0}))} + \|\pi\|_{L^2(B_{\rho_0}(\zeta_{i,0}))}\right) + C\epsilon^{\alpha^k}\left(\|w\|_{H^3(\tilde{\Omega}_i)} + \|\pi\|_{H^2(\tilde{\Omega}_i)}\right), \end{aligned} \tag{23}$$

where $\tilde{\Omega}_i \subset \Omega$ is an open set such that $\Gamma \subset \partial\tilde{\Omega}_i$ and $\mathcal{C}_i \subset \tilde{\Omega}_i$ and such that $(w, \pi) \in H^3(\tilde{\Omega}_i) \times H^2(\tilde{\Omega}_i)$.

We consider

$$\mathcal{C}'_i = \{x \in \mathbb{R}^d / (x - x_i) \cdot \xi_i > |x - x_i| \cos\theta' \text{ et } |x - x_i| \leq R_0\},$$

with

$$\theta' = \arcsin(t \sin\theta). \tag{24}$$

The parameter t belongs to $(0, 1)$ and will be specified later on. Note that we have $\mathcal{C}'_i \subset \mathcal{C}_i \subset \Omega$. We denote by $\zeta_{i,0} = x_i + \frac{R_0}{2}\xi_i$, $d_0 = |\zeta_{i,0} - x_i| = \frac{R_0}{2}$ and $\rho_0 = d_0 \sin\theta'$. For $k \in \mathbb{N}^*$, we define the sequence of balls by induction:

$$\zeta_{i,k+1} = x_i + \mu|\zeta_{i,k} - x_i|\xi_i, \quad d_{k+1} = |\zeta_{i,k+1} - x_i|, \quad \rho_{k+1} = d_{k+1}\sin(\theta'),$$

with

$$\mu = \frac{1 - \sin\theta'}{1 + \sin\theta'} \Longleftrightarrow \sin\theta' = \frac{1 - \mu}{1 + \mu}. \tag{25}$$

We refer to Figure 2 for an illustration of this construction. This construction implies that $d_{k+1} = \mu d_k$ and $\rho_{k+1} = \mu\rho_k$. We choose the parameter t involved in (24) such that $B_{8\rho_k}(\zeta_{i,k}) \subset \mathcal{C}_i \subset \Omega$ for all $k \in \mathbb{N}$, that is to say, using (24), $8\rho_k = 8d_k \sin\theta' \leq \sin\theta d_k \Leftrightarrow t \leq \frac{1}{8}$.

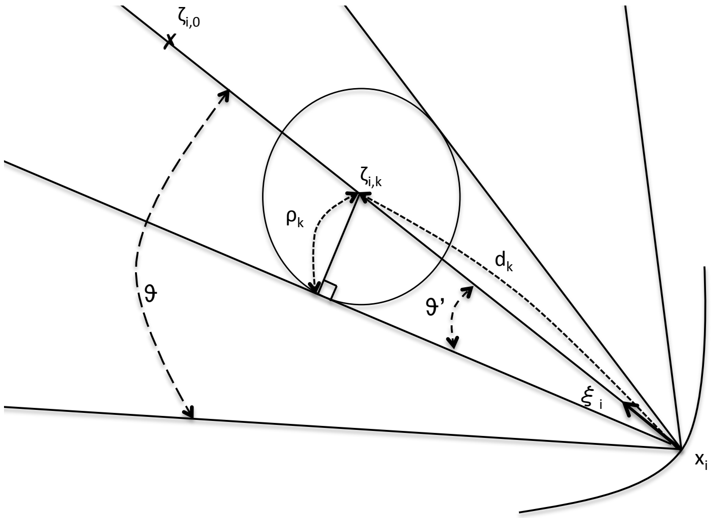

Figure 2: Figure illustrating the construction of the sequence of balls with decreasing radius and whose center is approaching x_i.

Next, we apply Lemma 3.6 with $\rho = \rho_0$ and $q = \zeta_{i,0}$: there exists $C > 0$ and $\alpha > 0$ such that

$$\begin{aligned}&\|w\|_{H^1(B_{3\rho_0}(\zeta_{i,0}))} + \|\pi\|_{L^2(B_{3\rho_0}(\zeta_{i,0}))}\\ &\qquad \leq C\left(\|w\|_{H^1(B_{\rho_0}(\zeta_{i,0}))} + \|\pi\|_{L^2(B_{\rho_0}(\zeta_{i,0}))}\right)^{\alpha}\left(\|w\|_{H^1(B_{8\rho_0}(\zeta_{i,0}))} + \|\pi\|_{L^2(B_{8\rho_0}(\zeta_{i,0}))}\right)^{1-\alpha}.\end{aligned}$$

Taking into account Lemma 3.8, since $\rho_{k-1} = \mu^{k-1}\rho_0$, it follows that, for all $k \in \mathbb{N}^*$:

$$\begin{aligned}&\|w\|_{H^1(B_{3\rho_{k-1}}(\zeta_{i,k-1}))} + \|\pi\|_{L^2(B_{3\rho_{k-1}}(\zeta_{i,k-1}))}\\ &\qquad \leq \frac{C}{\mu^{k-1}}\left(\|w\|_{H^1(B_{\rho_{k-1}}(\zeta_{i,k-1}))} + \|\pi\|_{L^2(B_{\rho_{k-1}}(\zeta_{i,k-1}))}\right)^{\alpha} \times\\ &\qquad\qquad \left(\|w\|_{H^1(B_{8\rho_{k-1}}(\zeta_{i,k-1}))} + \|\pi\|_{L^2(B_{8\rho_{k-1}}(\zeta_{i,k-1}))}\right)^{1-\alpha}. \end{aligned} \tag{26}$$

By construction, we have:

$$B_{\rho_k}(\zeta_{i,k}) \subseteq B_{3\rho_{k-1}}(\zeta_{i,k-1}). \tag{27}$$

Indeed, we have $\zeta_{i,k} - \zeta_{i,k-1} = \mu(d_{k-1} - d_{k-2})\xi_i = \dfrac{\mu}{\sin\theta'}(\rho_{k-1} - \rho_{k-2})\xi_i = -\dfrac{\mu}{\sin\theta'}(1-\mu)\rho_{k-2}\xi_i$ and using (24), we obtain $\zeta_{i,k} - \zeta_{i,k-1} = -\mu(1+\mu)\rho_{k-2}\xi_i$. Then, if $x \in B_{\rho_k}(\zeta_{i,k})$, we have:

$$|x - \zeta_{i,k-1}| \leq |x - \zeta_{i,k}| + |\zeta_{i,k-1} - \zeta_{i,k}| \leq \rho_k + \mu(1+\mu)\rho_{k-2} \leq 3\rho_{k-1}.$$

By combining (26) and (27) we deduce that there exists $C > 0$ and $0 < \alpha < 1$ such that for all $k \in \mathbb{N}^*$, we have:

$$\begin{aligned}&\|w\|_{H^1(B_{\rho_k}(\zeta_{i,k}))} + \|\pi\|_{L^2(B_{\rho_k}(\zeta_{i,k}))}\\ &\leq \frac{C}{\mu^{k-1}}\left(\|w\|_{H^1(B_{\rho_{k-1}}(\zeta_{i,k-1}))} + \|\pi\|_{L^2(B_{\rho_{k-1}}(\zeta_{i,k-1}))}\right)^{\alpha}\left(\|w\|_{H^1(B_{8\rho_{k-1}}(\zeta_{i,k-1}))} + \|\pi\|_{L^2(B_{8\rho_{k-1}}(\zeta_{i,k-1}))}\right)^{1-\alpha}.\end{aligned} \tag{28}$$

Let $\tilde{\Omega}_i \subset \Omega$ be an open set such that $\Gamma \subset \partial\tilde{\Omega}_i$, $\mathcal{C}_i \subset \tilde{\Omega}_i$ and such that $(w,\pi) \in H^3(\tilde{\Omega}_i) \times H^2(\tilde{\Omega}_i)$. Note that such an open set exists thanks to local regularity results stated in Subsection 2.2. We have, for all $k \in \mathbb{N}^*$:

$$\|w\|_{H^1(B_{8\rho_{k-1}}(\zeta_{i,k-1}))} + \|\pi\|_{L^2(B_{8\rho_{k-1}}(\zeta_{i,k-1}))} \leq \|w\|_{H^3(\tilde{\Omega}_i)} + \|\pi\|_{H^2(\tilde{\Omega}_i)}.$$

Thus, we can rewrite inequality (28) as:

$$\begin{aligned}&\|w\|_{H^1(B_{\rho_k}(\zeta_{i,k}))} + \|\pi\|_{L^2(B_{\rho_k}(\zeta_{i,k}))}\\ &\quad\leq \frac{C}{\mu^{k-1}} \left(\|w\|_{H^1(B_{\rho_{k-1}}(\zeta_{i,k-1}))} + \|\pi\|_{L^2(B_{\rho_{k-1}}(\zeta_{i,k-1}))}\right)^{\alpha} \left(\|w\|_{H^3(\tilde{\Omega}_i)} + \|\pi\|_{H^2(\tilde{\Omega}_i)}\right)^{1-\alpha}.\end{aligned}$$

We now apply Lemma 3.2 with $\beta_k = \|w\|_{H^1(B_{\rho_k}(\zeta_{i,k}))} + \|\pi\|_{L^2(B_{\rho_k}(\zeta_{i,k}))}$ and $A = C\left(\|w\|_{H^3(\tilde{\Omega}_i)} + \|\pi\|_{H^2(\tilde{\Omega}_i)}\right)$, to obtain:

$$\begin{aligned}&\|w\|_{H^1(B_{\rho_k}(\zeta_{i,k}))} + \|\pi\|_{L^2(B_{\rho_k}(\zeta_{i,k}))}\\ &\quad\leq \frac{1}{\mu^{\iota}} \left(\|w\|_{H^1(B_{\rho_0}(\zeta_{i,0}))} + \|\pi\|_{L^2(B_{\rho_0}(\zeta_{i,0}))}\right)^{\alpha^k} \left(C\left(\|w\|_{H^3(\tilde{\Omega}_i)} + \|\pi\|_{H^2(\tilde{\Omega}_i)}\right)\right)^{1-\alpha^k},\end{aligned}$$

where $\iota = \displaystyle\sum_{j=1}^{k-1} j\alpha^{k-1-j}$. Note that $\iota \leq (k-1)\left(\displaystyle\sum_{j=0}^{k-2}\alpha^j\right) \leq \dfrac{k-1}{1-\alpha}$. To summarize, the following inequality is satisfied:

$$\begin{aligned}&\|w\|_{H^1(B_{\rho_k}(\zeta_{i,k}))} + \|\pi\|_{L^2(B_{\rho_k}(\zeta_{i,k}))}\\ &\quad\leq \frac{1}{\mu^{\frac{k-1}{1-\alpha}}} \left(\|w\|_{H^1(B_{\rho_0}(\zeta_{i,0}))} + \|\pi\|_{L^2(B_{\rho_0}(\zeta_{i,0}))}\right)^{\alpha^k} \left(C(\|w\|_{H^3(\tilde{\Omega}_i)} + \|\pi\|_{H^2(\tilde{\Omega}_i)})\right)^{1-\alpha^k}.\end{aligned}$$

Let $\epsilon > 0$. By Lemma 3.3 with $A = \|w\|_{H^1(B_{\rho_0}(\zeta_{i,0}))} + \|\pi\|_{L^2(B_{\rho_0}(\zeta_{i,0}))}$, $B = C(\|w\|_{H^3(\tilde{\Omega}_i)} + \|\pi\|_{H^2(\tilde{\Omega}_i)})$, $C_1 = \dfrac{1}{\mu^{\frac{k-1}{1-\alpha}}}$, and $D = \|w\|_{H^1(B_{\rho_k}(\zeta_{i,k}))} + \|\pi\|_{L^2(B_{\rho_k}(\zeta_{i,k}))}$, we obtain:

$$\begin{aligned}&\|w\|_{H^1(B_{\rho_k}(\zeta_{i,k}))} + \|\pi\|_{L^2(B_{\rho_k}(\zeta_{i,k}))}\\ &\quad\leq \frac{c_k}{\epsilon}\left(\|w\|_{H^1(B_{\rho_0}(\zeta_{i,0}))} + \|\pi\|_{L^2(B_{\rho_0}(\zeta_{i,0}))}\right) + C\epsilon^{s_k}\left(\|w\|_{H^3(\tilde{\Omega}_i)} + \|\pi\|_{H^2(\tilde{\Omega}_i)}\right),\end{aligned}$$

with $s_k = \dfrac{\alpha^k}{1-\alpha^k}$ and $c_k = \left(\dfrac{1}{\mu^{\frac{k-1}{1-\alpha}}}\right)^{1/\alpha^k}$. Note that $s_k \geq \alpha^k$ implies, for $0 < \epsilon < 1$ that:

$$\begin{aligned}&\|w\|_{H^1(B_{\rho_k}(\zeta_{i,k}))} + \|\pi\|_{L^2(B_{\rho_k}(\zeta_{i,k}))}\\ &\quad\leq \frac{e^{\frac{C}{\alpha^k}\log\left(\frac{1}{\mu^{k-1}}\right)}}{\epsilon}\left(\|w\|_{H^1(B_{\rho_0}(\zeta_{i,0}))} + \|\pi\|_{L^2(B_{\rho_0}(\zeta_{i,0}))}\right) + C\epsilon^{\alpha^k}\left(\|w\|_{H^3(\tilde{\Omega}_i)} + \|\pi\|_{H^2(\tilde{\Omega}_i)}\right).\end{aligned}$$

Let us remark that the previous inequality is clearly satisfied for $\epsilon \geq 1$ by continuity of the injection $H^1(\tilde{\Omega}_i) \hookrightarrow H^1(B_{\rho_k}(\zeta_{i,k}))$. Thus, we obtain inequality (23).

Step 2 : Combining local Hölder regularity of the solution and inequality (23), and then optimizing the resulting inequality, we will obtain the desired Lipschitz stability estimate (20).

Let us recall that thanks to Corollary 2.4, there exists $R > 0$ and $0 < \beta < 1$ such that $(w, \pi) \in \mathcal{C}^{1,\beta}(\overline{B_R(x_i) \cap \Omega}) \times \mathcal{C}^{0,\beta}(\overline{B_R(x_i) \cap \Omega})$ for all $i = 1, \dots, N$.

Let $0 < \epsilon' < \min(R, 1) = \epsilon'_0$. If $d_k + \rho_k < \epsilon'$, that is to say $\mu^k(d_0 + \rho_0) < \epsilon'$, we have $B_{\rho_k}(\zeta_{i,k}) \subset B_{\epsilon'}(x_i)$. Let $k_0 = k_0(\epsilon')$ be the smallest k which satisfies this inequality. We have:

$$\left| \frac{\log((d_0 + \rho_0)/\epsilon')}{\log(1/\mu)} \right| \le k_0 < \left| \frac{\log((d_0 + \rho_0)/\epsilon')}{\log(1/\mu)} \right| + 1. \tag{29}$$

Since (w, π) is solution of system (22), we obtain, using the boundary condition on Γ_i:

$$\frac{|q_i^1 - q_i^2|}{\sum_{j=1}^N |q_j^1 - q_j^2|} |u_2(x_i)| \le \left| \frac{\partial w}{\partial n}(x_i) \right| + |\pi(x_i)| + |q_i^1||w(x_i)| \le |\nabla w(x_i)| + |\pi(x_i)| + R_M |w(x_i)|. \tag{30}$$

Let $y_i \in B_{\rho_{k_0}}(\zeta_{i,k_0})$. Using the Hölder regularity of w, ∇w and π on $B_{\rho_{k_0}}(\zeta_{i,k_0}) \subset B_{\epsilon'}(x_i) \subset B_R(x_i)$, we have:

$$\frac{|q_i^1 - q_i^2|}{\sum_{j=1}^N |q_j^1 - q_j^2|} |u_2(x_i)| \le |\nabla w(y_i)| + |\pi(y_i)| + R_M |w(y_i)| + C|x_i - y_i|^\beta.$$

Moreover, since $|u_2(x_i)| > m$, we obtain that, for all y_i in $B_{\rho_{k_0}}(\zeta_{i,k_0})$,

$$\frac{|q_i^1 - q_i^2|}{\sum_{j=1}^N |q_j^1 - q_j^2|} m \le |\nabla w(y_i)| + |\pi(y_i)| + R_M |w(y_i)| + C{\epsilon'}^\beta.$$

Let us denote by ω_d the volume of the unit ball in $\mathbb{R}^d$. By integrating in L^2 norm the previous inequality in $B_{\rho_{k_0}}(\zeta_{i,k_0})$, we obtain:

$$\frac{|q_i^1 - q_i^2|}{\sum_{j=1}^N |q_j^1 - q_j^2|} \le \frac{C(m, R_M)}{\omega_d^{\frac{1}{2}} \rho_{k_0}^{\frac{d}{2}}} \left(\|w\|_{H^1\left(B_{\rho_{k_0}}(\zeta_{i,k_0})\right)} + \|\pi\|_{L^2\left(B_{\rho_{k_0}}(\zeta_{i,k_0})\right)} \right) + C{\epsilon'}^\beta.$$

The previous inequality together with (23) yields to the existence of $0 < \alpha < 1$, $0 < \mu < 1$ such that for all $i = 1, \dots, N$ the following estimate holds for all $\epsilon > 0$

$$\begin{aligned} \frac{|q_i^1 - q_i^2|}{\sum_{j=1}^N |q_j^1 - q_j^2|} \le \frac{C(m, R_M)}{\rho_{k_0}^{\frac{d}{2}}} \frac{e^{\frac{C}{\alpha^{k_0}} \log\left(\frac{1}{\mu^{k_0 - 1}}\right)}}{\epsilon} \left(\|w\|_{H^1(B_{\rho_0}(\zeta_{i,0}))} + \|\pi\|_{L^2(B_{\rho_0}(\zeta_{i,0}))} \right) \\ + \frac{C(m, R_M)}{\rho_{k_0}^{\frac{d}{2}}} \epsilon^{\alpha^{k_0}} \left(\|w\|_{H^3(\tilde{\Omega}_i)} + \|\pi\|_{H^2(\tilde{\Omega}_i)} \right) + C{\epsilon'}^\beta. \end{aligned}$$

By summing up the previous inequality for $i = 1$, ..., N, we obtain:

$$\begin{aligned} 1 \le \frac{C(m, R_M)}{\rho_{k_0}^{\frac{d}{2}}} \frac{e^{\frac{C}{\alpha^{k_0}} \log\left(\frac{1}{\mu^{k_0 - 1}}\right)}}{\epsilon} \sum_{i=1}^N \left(\|w\|_{H^1(B_{\rho_0}(\zeta_{i,0}))} + \|\pi\|_{L^2(B_{\rho_0}(\zeta_{i,0}))} \right) \\ + \frac{C(m, R_M, N)}{\rho_{k_0}^{\frac{d}{2}}} \epsilon^{\alpha^{k_0}} \left(\|w\|_{H^3(\tilde{\Omega})} + \|\pi\|_{H^2(\tilde{\Omega})} \right) + C(N){\epsilon'}^\beta, \end{aligned}$$

where $\tilde{\Omega} = \bigcup_{i=1}^{N} \tilde{\Omega}_i$. Moreover, $\mu^{k_0-1}(d_0+\rho_0) \geq \epsilon'$ and $\rho_{k_0} = \mu^{k_0}\rho_0$ imply that $\rho_{k_0} \geq \mu \dfrac{\rho_0}{d_0+\rho_0}\epsilon'$. It follows:

$$1 \leq \frac{C(m,R_M)}{\epsilon'^{\frac{d}{2}}} \frac{e^{\frac{C}{\alpha^{k_0}}\log\left(\frac{1}{\mu^{k_0-1}}\right)}}{\epsilon} \sum_{i=1}^{N} \left(\|w\|_{H^1(B_{\rho_0}(\zeta_{i,0}))} + \|\pi\|_{L^2(B_{\rho_0}(\zeta_{i,0}))} \right)$$
$$+ C(m,R_M,N) \left(\|w\|_{H^3(\tilde{\Omega})} + \|\pi\|_{H^2(\tilde{\Omega})} + C(N) \right) \left(\frac{\epsilon^{\alpha^{k_0}}}{\epsilon'^{\frac{d}{2}}} + \epsilon'^{\beta} \right). \quad (31)$$

Let us denote by

$$E = \left(\|w\|_{H^3(\tilde{\Omega})} + \|\pi\|_{H^2(\tilde{\Omega})} + C(N) \right). \quad (32)$$

We simplify the last term in the right hand-side in (31) by choosing $\epsilon > 0$ such that $\dfrac{\epsilon^{\alpha^{k_0}}}{\epsilon'^{\frac{d}{2}}} = \epsilon'^{\beta}$. Since $\alpha^{k_0} < 1$, we obtain, for all ϵ' small enough

$$1 \leq C(m,R_M,N) \frac{e^{\frac{C}{\alpha^{k_0}}\log\left(\frac{1}{\mu^{k_0-1}}\right)}}{\epsilon'^{(\beta+d)/\alpha^{k_0}}} \sum_{i=1}^{N} \left(\|w\|_{H^1(B_{\rho_0}(\zeta_{i,0}))} + \|\pi\|_{L^2(B_{\rho_0}(\zeta_{i,0}))} \right) + C(m,R_M,N) E \epsilon'^{\beta}. \quad (33)$$

Since $\dfrac{1}{\alpha^{k_0}} = e^{k_0 \log(1/\alpha)}$, we obtain, using (29):

$$\frac{1}{\alpha^{k_0}} < e^{\log(1/\alpha)\left(\frac{\log((d_0+\rho_0)/\epsilon')}{\log(1/\mu)} + 1 \right)} = \frac{1}{\alpha} \left(\frac{d_0+\rho_0}{\epsilon'} \right)^{\gamma_0},$$

where $\gamma_0 = \dfrac{\log(1/\alpha)}{\log(1/\mu)}$. Furthermore, since $\dfrac{1}{\mu^{k_0-1}} \leq \dfrac{d_0+\rho_0}{\epsilon'}$ by definition of k_0, we have

$$\log\left(\frac{1}{\mu^{k_0-1}} \right) < \log\left(\frac{d_0+\rho_0}{\epsilon'} \right).$$

Then,

$$\frac{e^{\frac{c}{\alpha^{k_0}}\log\left(\frac{1}{\mu^{k_0-1}}\right)}}{\epsilon'^{(\beta+d)/\alpha^{k_0}}} = e^{\frac{1}{\alpha^{k_0}}\left(c\log\left(\frac{1}{\mu^{k_0-1}}\right) + (\beta+d)\log\left(\frac{1}{\epsilon'}\right) \right)}$$
$$\leq e^{\frac{1}{\alpha}\left(\frac{d_0+\rho_0}{\epsilon'}\right)^{\gamma_0}\left(c\log\left(\frac{d_0+\rho_0}{\epsilon'}\right) + (\beta+d)\log\left(\frac{1}{\epsilon'}\right) \right)} \leq e^{\frac{C}{\epsilon'^{\gamma_0}}\log\left(\frac{1}{\epsilon'}\right)}. \quad (34)$$

To summarize, for $\gamma > \gamma_0$ and for $0 < \epsilon' < \epsilon'_0$, we obtain from (33) and (34):

$$1 \leq C(m,R_M,N) \left(e^{C/\epsilon'^{\gamma}} \sum_{i=1}^{N} \left(\|w\|_{H^1(B_{\rho_0}(\zeta_{i,0}))} + \|\pi\|_{L^2(B_{\rho_0}(\zeta_{i,0}))} \right) + E\epsilon'^{\beta} \right).$$

By denoting $\dfrac{1}{s} = \epsilon'^{\gamma}$, it can be rewritten for all $s > s_0 = \dfrac{1}{\epsilon_0'^{\gamma}}$ as:

$$1 \leq C(m,R_M,N) \left(e^{Cs} \sum_{i=1}^{N} \left(\|w\|_{H^1(B_{\rho_0}(\zeta_{i,0}))} + \|\pi\|_{L^2(B_{\rho_0}(\zeta_{i,0}))} \right) + E\left(\frac{1}{s}\right)^{\beta/\gamma} \right).$$

By applying Corollary 3.5, there exist $C(M_1, m, R_M, N) > 0$ and $0 < \delta < 1$ such that for all $s > s_0$:

$$1 \leq C(M_1, m, R_M, N)\left(e^{Cs}\left(\|w\|_{L^2(\Gamma)} + \|\pi\|_{L^2(\Gamma)} + \left\|\frac{\partial w}{\partial n}\right\|_{L^2(\Gamma)} + \left\|\frac{\partial \pi}{\partial n}\right\|_{L^2(\Gamma)}\right)^{\delta} + E\left(\frac{1}{s}\right)^{\beta/\gamma}\right). \tag{35}$$

Note that the previous inequality remains true for $0 < s < s_0$, eventually by increasing the constant $C(N)$ involved in (32).

We now look for the lower bound of this inequality with respect to s. We denote by

$$\Lambda = \left(\|w\|_{L^2(\Gamma)} + \|\pi\|_{L^2(\Gamma)} + \left\|\frac{\partial w}{\partial n}\right\|_{L^2(\Gamma)} + \left\|\frac{\partial \pi}{\partial n}\right\|_{L^2(\Gamma)}\right)^{\delta}. \tag{36}$$

and $f(s) = e^{Cs}\Lambda + \tilde{d}E\left(\frac{1}{s}\right)^{\beta/\gamma}$, with $\tilde{d} \geq 1$. Let us study the function f in $\mathbb{R}_+^*$. We have:

$$\begin{cases} \lim_{s\to 0} f(s) = +\infty, \\ \lim_{s\to\infty} f(s) = +\infty. \end{cases}$$

So since f is continuous on $\mathbb{R}_*^+$, f reaches its minimum at a point $s_0 > 0$. At this point,

$$f'(s_0) = 0 \Leftrightarrow \Lambda = \frac{E\tilde{d}\beta}{C\gamma}\frac{e^{-cs_0}}{s_0^{\frac{\beta}{\gamma}+1}}, \text{ thus } f(s_0) = \frac{\beta}{C\gamma}\frac{E\tilde{d}}{s_0^{\frac{\beta}{\gamma}+1}} + \frac{E\tilde{d}}{s_0^{\frac{\beta}{\gamma}}}.$$

Hence, (35) leads to:

$$1 \leq \frac{E\tilde{d}C(M_1, m, R_M, N)}{s_0^{\lambda}}\left(\frac{\beta}{c\gamma} + 1\right), \tag{37}$$

where $\lambda = \frac{\beta}{\gamma}$ if $s_0 \geq 1$ and $\lambda = 1 + \frac{\beta}{\gamma}$ otherwise. But:

$$\frac{1}{\Lambda} = \frac{C\gamma}{E\tilde{d}\beta}s_0^{\frac{\beta}{\gamma}+1}e^{cs_0} \leq \frac{C\gamma}{E\tilde{d}\beta}e^{(\frac{\beta}{\gamma}+1+c)s_0},$$

which can be written as follows:

$$\frac{1}{s_0} \leq \frac{\frac{\beta}{\gamma} + 1 + C}{\ln\left(\frac{E\tilde{d}\beta}{C\gamma\Lambda}\right)},$$

for all $\tilde{d}$ large enough. Taking into account (37), it leads to:

$$1 \leq \frac{E\tilde{d}C(M_1, m, R_M, N)}{\ln\left(\frac{E\tilde{d}\beta}{C\gamma\Lambda}\right)^{\lambda}}.$$

Thanks to local regularity stated in Subsection 2.2, we know that there exits a constant $C(M_1, R_M) > 0$ such that:

$$\|w\|_{H^3(\tilde{\Omega})} + \|\pi\|_{H^2(\tilde{\Omega})} \leq C(M_1, R_M),$$

which, remembering the definition (32) of E, leads to $E \leq C(M_1, R_M, N)$. Thus, by studying the variation of the function $f_y(x) = \dfrac{x}{(ln(\frac{x}{y}))^\lambda}$ on $(y, +\infty)$, for $y = \dfrac{C\gamma\Lambda}{\tilde{d}\beta}$ we obtain the existence of two positive constants $C_1(M_1, m, R_M, N)$ and $C_2(M_1, R_M, N)$ such that

$$1 \leq \frac{C_1(M_1, m, R_M, N)}{\left(\ln\left(\frac{C_2(M_1, R_M, N)}{\Lambda}\right)\right)^\lambda}.$$

Remembering the definition (36) of Λ, the previous inequality is equivalent to

$$\|w\|_{L^2(\Gamma)} + \|\pi\|_{L^2(\Gamma)} + \left\|\frac{\partial w}{\partial n}\right\|_{L^2(\Gamma)} + \left\|\frac{\partial \pi}{\partial n}\right\|_{L^2(\Gamma)} \geq C(M_1, m, R_M, N).$$

By replacing (w, π) by (21), we obtain:

$$\sum_{j=1}^{N} |q_j^1 - q_j^2| \leq$$
$$C(m, R_M, M_1, N)\left(\|u_1 - u_2\|_{L^2(\Gamma)} + \|p_1 - p_2\|_{L^2(\Gamma)} + \left\|\frac{\partial u_1}{\partial n} - \frac{\partial u_2}{\partial n}\right\|_{L^2(\Gamma)} + \left\|\frac{\partial p_1}{\partial n} - \frac{\partial p_2}{\partial n}\right\|_{L^2(\Gamma)}\right)$$

which concludes the proof of Theorem 4.1, since $\dfrac{\partial u_1}{\partial n} - \dfrac{\partial u_2}{\partial n} = (p_1 - p_2)n$ on Γ. $\square$

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ISSN 0249-6399